Education Insights

Article

Research on the Teaching of Chapter Opening Lessons in High School Mathematics from the Perspective of Problem Posing - Taking "Lines and Circles" as an Example

Zeyu Wei [1,*] and Yinuo He [1]

[1] School of Teacher Education, Nanjing Normal University, Nanjing, Jiangsu, 210023, China
* Correspondence: Zeyu Wei, School of Teacher Education, Nanjing Normal University, Nanjing, Jiangsu, 210023, China

Abstract: Mathematical problem posing has emerged as a prominent topic in the international field of mathematics education. China attaches great importance to this, emphasizing the cultivation of students' ability to propose mathematical problems and highlighting the requirement for the structuring of mathematical content in the Mathematics Curriculum Standards for Compulsory Education (2022 version). Taking the chapter opening lesson of "Equations of Lines and Circles" in high school mathematics as an example, this paper explores the teaching design and practical paths of chapter opening lessons from the perspective of problem posing. The study constructs a four-dimensional holistic questioning scaffold of "time-space-external domain-internal domain" to guide students in understanding "why to learn" through holistic questioning; further employs the "What-If-Not" strategy to build a systematic questioning scaffold, enabling students to explore various attributes of the mathematical objects being studied and clarify "what to learn"; finally, guides students to actively propose high-order mathematical problems based on the extracted attributes, thereby realizing the transformation of teaching paradigm from passive acceptance to active inquiry.

Keywords: chapter opening lesson; Lines and Circles; problem posing; teaching design; What-If-Not Strategy

Published: 19 December 2025

Copyright: © 2025 by the authors. Submitted for possible open access publication under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Mathematical problem posing has emerged as an important topic in the international field of mathematics education and has attracted sustained attention from scholars across different educational systems. Research has increasingly recognized problem posing as a meaningful cognitive activity that supports mathematical understanding, creativity, and higher-order thinking. Nevertheless, despite its growing theoretical significance, mathematical problem posing has not yet been widely incorporated into the mathematics curriculum standards and textbooks of many countries. In some national curricula, such as those of the Philippines and England, relevant content is largely absent; in others, including Germany and Singapore, only limited references can be found. By contrast, South Korea has gradually expanded curriculum content related to mathematical problem posing, while the curriculum standards of mainland China explicitly emphasize its importance [1]. The Mathematics Curriculum Standards for Compulsory Education (2022 edition) clearly propose the cultivation of students' ability to pose mathematical problems as an essential educational objective [2]. From a comparative perspective, the explicit emphasis placed on mathematical problem posing at the curriculum policy level in China reflects a certain degree of foresight. Research in this area therefore not only serves the needs of local mathematics education practice but also contributes constructive experience that may inform broader international discussions on mathematics education reform.

At the same time, the new curriculum standards advocate a restructuring of mathematical content through unit-based holistic teaching. This approach calls for a shift away from an excessive focus on lesson-by-lesson instructional design toward the promotion of students' integrated understanding of mathematical knowledge and the development of core competencies [2,3]. Within this instructional orientation, the chapter opening lesson has gradually gained attention as a distinctive and important teaching form. Such lessons are expected to guide subsequent unit learning, clarify learning objectives, and support students in constructing coherent knowledge structures [4]. However, classroom practice indicates that chapter opening lessons often encounter practical difficulties. Teachers may hold misconceptions about their functions, or lack concrete implementation strategies, which prevents these lessons from fully realizing their intended instructional value [5]. As a result, the potential of chapter opening lessons to support meaningful learning and conceptual organization remains underdeveloped in many teaching contexts.

Previous studies suggest that developing transferable teaching cases is an effective way to promote the classroom implementation of problem posing [6]. Teaching cases can provide concrete references for teachers, reduce the difficulty of instructional design, and facilitate the dissemination of effective pedagogical approaches. On this basis, the present study aims to explore the adaptability of applying problem posing theory to the teaching of chapter opening lessons. Taking the chapter *Equations of Lines and Circles* from the People's Education Press senior high school elective mathematics textbook (Volume 1) as an illustrative example, this study examines how problem posing methods can be systematically integrated into chapter opening lesson design, and seeks to clarify practical instructional pathways for their application in classroom teaching.

2. Value Alignment between Problem Posing Theory and Chapter Opening Lesson Teaching

Problem posing and chapter opening lessons are highly aligned in teaching logic. Bruner's theory of constructivism emphasizes the importance of mastering the basic structure of a discipline, while the chapter opening lesson aims to help students initially establish a knowledge map of the chapter through an overall framework [7]. In this process, high-quality questions can stimulate cognitive interest and deepen knowledge understanding. Problem posing strategies such as the "What-If-Not" method serve as cognitive "scaffolds" that are both holistic and systematic, which are particularly consistent with the core demands of chapter opening lessons-"why to learn, what to learn, and how to learn"-thereby guiding students to actively explore the basic framework of the chapter content.

This study draws on the Holistic Questioning and What-If-Not Strategy proposed by American scholars Brown and Walter in their book The Art of Problem Posing, and designs a holistic questioning framework and a systematic questioning framework based on these strategies. With the support of this framework, teachers can design high-quality problem scenarios to guide students in understanding "why to learn" through holistic questioning, extracting attributes of mathematical objects and clarifying "what to learn" through systematic questioning, and constructing high-order problems and attempting to solve them using mathematical problem posing thinking, thereby addressing the in-depth question of "how to learn".

2.1. Holistic Questioning to Achieve the Teaching Goal of "Why to Learn" in Chapter Opening Lessons

As the introductory lesson of chapter teaching, the chapter opening lesson is the starting point of unit cognition. Its core teaching objective is to guide students to clearly understand the generation logic and exploration value of the research problems in the unit, and further comprehend the background source and research significance of the knowledge to be learned. The "Holistic Questioning" strategy proposed by American scholars Brown and Walter in their book The Art of Problem Posing requires teachers to construct specific classroom scenarios, guide students to participate in activities in an implicit and interesting manner, naturally present the core problems to be explored in the process of activities, and then encourage students to put forward holistic questions about classroom knowledge from a macro perspective, ultimately deepening their reflection on the significance of the activities. The core logic of this strategy is highly consistent with the teaching objectives of chapter opening lessons. From a teaching practice perspective, teachers need to fully rely on textbook resources, focus on pre-chapter images and pre-chapter texts, preset guiding basic questions around these two types of content, and gradually inspire students to actively ask questions based on textbook materials in class, thereby better realizing the teaching goal of "why to learn" in chapter opening lessons.

Taking the chapter opening lesson of *"Equations of Lines and Circles"* (Chapter 2 of *People's Education Press Elective Volume 1*) as an example, this study constructs a problem scaffold for holistic questioning in chapter opening lessons from four categories—"time dimension", "space dimension", "external domain of mathematical objects", and "internal domain of mathematical objects"—based on the pre-chapter texts and pre-chapter images (see Table 1), providing teachers with richer ideas for conducting chapter opening lesson teaching.

Table 1. The Preface to the Chapter "Equations of Straight Lines and Circles".

Paragraph	The content of the chapter preface
The first paragraph	In previous geometry studies, we often investigated the shape, size, and positional relationships of geometric figures through methods such as intuitive perception, operational verification, speculative reasoning, and measurement calculation. This approach is commonly referred to as the synthetic method. In this chapter, we will use the coordinate method to study the properties of geometric figures. The coordinate method is the most fundamental research method in analytic geometry.
The second paragraph	Analytic geometry was founded in the 17th century by the French mathematicians René Descartes and Pierre de Fermat. Its basic concept and methodology involve establishing a correspondence between the fundamental elements of geometry-points-and the fundamental objects of algebra-numbers (ordered pairs or arrays)-through a coordinate system. On this basis, the equations of curves (loci of points) are established, thereby transforming geometric problems into algebraic problems. The properties of geometric figures are then studied through algebraic methods. The establishment of analytic geometry was a milestone in the history of mathematics, marking the entry of mathematics into the period of variable mathematics and laying the foundation for the creation of calculus.
The third paragraph	In this chapter, within the plane rectangular coordinate system, we will explore the geometric elements that determine the position of a line, establish the equations of lines, and use these equations to study the positional relationships between two lines, their intersection coordinates, and the distance from a point to a line. Similarly, by determining the geometric elements of a circle, we will

	establish the equation of a circle and then use it to investigate problems related to circles. Finally, we will apply the equations of lines and circles to solve some practical problems.

It should be noted that although students may spontaneously put forward some questions when viewing pre-chapter materials and conducting preliminary readings, teachers should still carefully preset a set of guiding questions before class to inspire students to propose more systematic and in-depth genuine questions from the above four dimensions, thereby better supporting their meaningful mathematical exploration.

2.1.1. Time Dimension

For this part, teachers can design questions around the background of the establishment of analytic geometry in the second paragraph of the pre-chapter text, guiding students to explore the core motivation for the birth of the coordinate method: Why did Descartes and Fermat in the 17th century break through the research limitations of the synthetic method? At the same time, teachers can connect the key context of mathematical development—the leap from static geometric research (synthetic method) to a variable-based mathematical system (coordinate method)—and examine how this transformation laid the foundation for the establishment of calculus. By exploring the historical logic of "method reform" and "disciplinary evolution," students can recognize that the invention of the coordinate method was an inevitable response to the bottleneck of geometric research, and further clarify the inheritance value of the knowledge in this chapter within the broader process of mathematical development. The corresponding problem scaffold for this time-dimension questioning is summarized in Table 2.

Table 2. Problem Scaffold for Holistic Questioning in the Time Dimension of Chapter Opening Lessons.

Category	Meaning	Example Questions
Time	From both diachronic and synchronic perspectives, guide students to explore the origin, development context, and contemporary application value of mathematical problems.	1.Why did Descartes, Fermat, and others invent the coordinate method in the 17th century? What were the common research methods used by mathematicians to study geometric problems before that? 2. The pre-chapter text mentions that mathematics entered the "variable mathematics" period. Were there other periods before that? Did each period have significant mathematical discoveries?

2.1.2. Space Dimension

In this teaching link, teachers can guide students to compare the development processes of Chinese and foreign mathematical cultures. On the one hand, students can perceive the diversity of world civilizations in mathematics learning; on the other hand, by studying Chinese mathematical history, they can deepen their identification with and confidence in their own culture. For example, teachers can guide students to ask in class: "Descartes and others invented the method of analytic geometry to study geometric problems. Were there similar discoveries in the history of Chinese mathematics?" In response to this question, teachers can supplement popular science on Chinese mathematical works such as *Zhou Bi Suan Jing* (*The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*) and *Jiu Zhang Suan Shu* (*Nine Chapters on the Art of Mathematics*). Furthermore, teachers can continue to guide students to think about the differences in the research on similar problems among different cultures. This culture-tracing teaching not only echoes the positioning of “analytic geometry serving practical problems” in the pre-chapter text but also allows students to naturally understand that “learning the knowledge in this chapter is an inevitable choice to inherit diverse mathematical cultures and decode the laws of real space” through comparisons between ancient and modern times, and between China and foreign countries. Cultural confidence also quietly grows in the collision of mathematical thinking. The problem scaffold corresponding to this space-dimension holistic questioning is presented in Table 3.

Table 3. Problem Scaffold for Holistic Questioning in the Space Dimension of Chapter Opening Lessons.

Category	Meaning	Example Questions
Space	Compare the cognitive differences of the same mathematical concept in different cultures or regions, and reveal the cultural diversity in the construction of mathematical knowledge and its impact on the way of problem posing.	1. Descartes and others invented the method of analytic geometry to study geometric problems. Were there similar discoveries in the history of Chinese mathematics? 2. How did ancient Chinese scholars study the positional relationship between lines and circles? Is this similar to the method of analytic geometry used by Descartes and others?

2.1.3. External Domain Dimension of Mathematical Objects

In this teaching link, teachers should guide students to break through the boundaries of mathematics, organically integrate mathematical objects or problems with fields such as art, technology, and philosophy, and explore the non-mathematical functions of mathematical knowledge. The pre-chapter image of the textbook (a cable-stayed bridge and a setting sun) provides a good entry point for this: teachers can first guide students to think about the role of the pre-chapter image and put forward the core question- “How do the images of the cable-stayed bridge and the setting sun connect with the equations of lines and circles to be learned later?” This image triggers students' thinking about interdisciplinary connections, and then guides them to associate and ask questions around the image, trying to establish multiple connections between visual elements and mathematical concepts, thereby stimulating their divergent thinking (Figure 1). The problem scaffold corresponding to this external-domain holistic questioning is presented in Table 4.

Figure 1. The Preface Figure of the Chapter "Equations of Straight Lines and Circles".

Table 4. Problem Scaffold for Holistic Questioning in the External Domain Dimension of Chapter Opening Lessons.

Category	Meaning	Example Questions
External Domain	Break through the boundaries of mathematics, combine mathematical objects or problems with fields such as art, technology, and philosophy, and explore their non-mathematical functions.	1.What is the connection between the pre-chapter image and the knowledge we are about to learn? 2. How is a cable-stayed bridge constructed? What knowledge of analytic geometry is required? 3. This picture is very beautiful, with the setting sun, bridge, and water surface being very harmonious. What photography techniques were used?

2.1.4. Internal Domain of Mathematical Objects

In this teaching link, teachers should focus on the method of analogy teaching. Analogy is a commonly used thinking method in middle school mathematics teaching, as well as an important way to propose conjectures and conduct mathematical research. Many mathematical contents have similar attributes, and new knowledge can be learned through analogy. In chapter opening lesson teaching, teachers can guide students to discover the similarities between the content to be learned and certain already learned knowledge, and predict from which aspects and by which methods this chapter will study new mathematical objects through analogy [7].

Guiding students to actively ask questions through analogy is not a simple "review of old knowledge". For example, in the chapter opening lesson of "Equations of Lines and Circles", teachers can let students independently discover the "subordinate relationship between linear functions and linear equations" and the "commonality in the derivation methods of equations of lines and circles" through questioning, thereby integrating new knowledge into the existing knowledge structure. This not only responds to the teaching objective of chapter opening lessons to "help students establish a global perspective" but also transforms the answer to "why to learn" from "told by teachers" to "independently constructed" through students' active questioning, deepening their understanding of knowledge background and logic (Table 5).

Table 5. Problem Scaffold for Holistic Questioning in the Internal Domain Dimension of Chapter Opening Lessons.

Category	Meaning	Example Questions
Internal Domain	Focus on the relevance within the mathematical knowledge system and establish logical bridges between different branches (such as algebra, geometry, and statistics).	1.This chapter tells us that lines and circles can be represented algebraically. Can all figures in the world be represented algebraically? 2. Will the research methods of linear functions, inverse proportional functions, and direct proportional functions we learned in junior high school be used in the study of this chapter?

2.2. Systematic Questioning to Implement the Content Construction of "What to Learn" in Chapter Opening Lessons

As a "content overview" of unit teaching, the chapter opening lesson aims to help students perceive the unit content as a whole and clarify learning objectives.

If students can be guided to independently complete this construction process through questioning, the teaching effect will be more ideal. Among the existing various questioning strategies, the What-If-Not method has a clear and operable implementation framework. Once students master its steps, they can use systematic methods to propose various questions and actively reconstruct their cognition of mathematical objects. Therefore, in chapter opening lessons, students can be guided to independently extract the attributes of the mathematical objects involved in the chapter and explore each category in a "negative" way-both realizing the independent generation and construction of chapter knowledge and effectively improving their ability to discover and propose problems.

The What-If-Not method is divided into five levels [8]. Teachers can design a stepped task chain to guide students to gradually master systematic skills of questioning, reconstructing, and expanding problems starting from a single knowledge point.

In this chapter opening lesson, starting from the students' existing knowledge that "the graph of a linear function is a straight line" (Level 0), guide students to extract two main attributes of a straight line (Level 1): (1) having a certain inclination angle; (2) passing through a specific point. On this basis, guide students to conduct generalized research on these two attributes at the algebraic level (negating the previously familiar "geometric" perspective) (Level 2), promote students to propose reasonable mathematical expressions (Level 3), and conduct detailed analysis on them (Level 4). In the final link of the analysis, students can be guided to use the same thinking to further analogize and explore the equation of a circle (Figure 2).

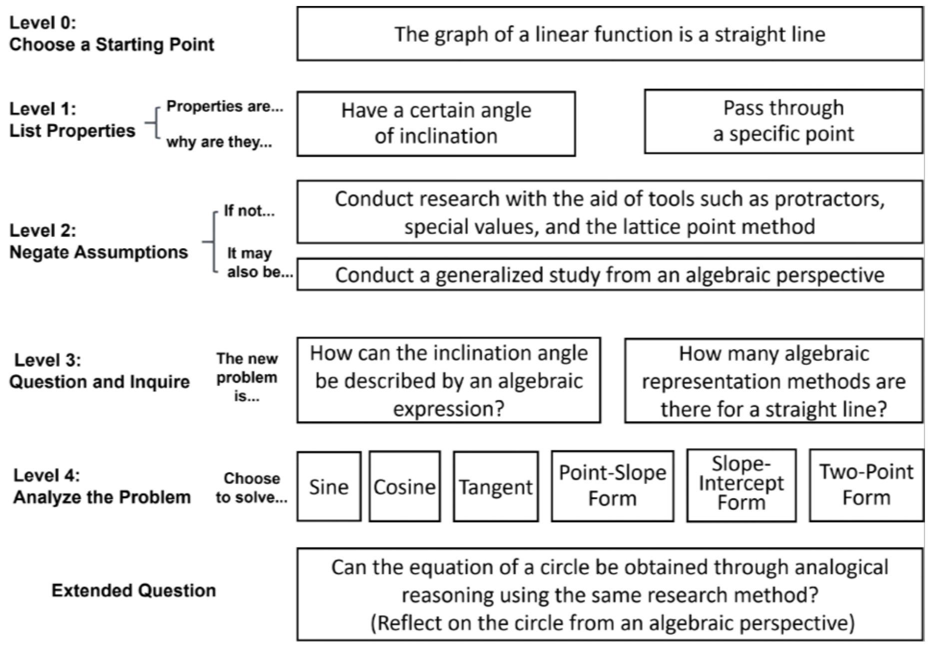

Figure 2. Systematic Questioning Framework for Chapter Opening Lessons.

2.2.1. Guiding Students to Extract Two Main Attributes of a Straight Line

In the teaching of the chapter opening lesson, starting from "why to learn", teachers guide students to move from geometric intuition to algebraic representation through problem scenarios. At the beginning of the teaching, teachers present numerous straight lines passing through a fixed point and guide students to observe and ask: "What are the differences between these straight lines?" Through visual comparison, students initially perceive that the "inclination degree" of the straight lines is different. Teachers further ask: "How can this difference be accurately described in a mathematical way?" Thereby stimulating students to propose the question: "Is there a quantity that can be used to distinguish straight lines in different directions?"

On this basis, teachers guide students to focus on the geometric attributes of straight lines and gradually extract two key characteristics: (1) directionality (i.e., degree of inclination) , (2) passing through a certain fixed point.

Through group discussions and teacher guidance, students realize that to uniquely determine a straight line, both its direction and the point it passes through must be clearly defined. In this process, students not only actively construct the mathematical necessity of "inclination angle" and "fixed point" but also lay a cognitive foundation for the subsequent introduction of slope and linear equations.

2.2.2. Generalized Research on Linear Equations from the Algebraic Perspective

After clarifying the geometric attributes of straight lines, teachers guide students to further ask: "Can the direction of a straight line be described algebraically?" Students attempt to express it using trigonometric function values of the inclination angle α. By calculating the trigonometric function values between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, students derive:

$$\sin\alpha = \frac{y_2 - y_1}{|PQ|},\ \cos\alpha = \frac{x_2 - x_1}{|PQ|},\ \tan\alpha = \frac{y_2 - y_1}{|x_2 - x_1|} \tag{1}$$

Teachers guide students to compare the three functions from three dimensions: uniqueness, simplicity, and sign stability: $\sin\alpha$ is not unique within $[0, \pi]$; $\cos\alpha$ is affected by the order of points and has unstable signs; $\tan\alpha$ has a simple form and is a constant value on the same straight line, possessing good algebraic properties.

Therefore, students unanimously agree that the slope (i.e., $\tan\alpha$) is the most appropriate algebraic representation. On this basis, teachers guide students to review linear functions $y = kx + b$ and prove that their graphs are straight lines through the constant slope k, realizing the initial integration of geometry and algebra.

Subsequently, teachers further guide students to ask: "How to represent an entire straight line using a fixed point and a slope?" Through the slope formula, students derive the point-slope form equation:

$$y - y_0 = k(x - x_0) \tag{2}$$

and further discuss the connections and differences between this form and other forms such as slope-intercept form and two-point form, understanding the geometric significance and applicable scenarios behind different representation methods, and completing the mathematical modeling process from specific to general and from geometry to algebra.

2.2.3. Analogical Reasoning to Obtain the Equation of a Circle

After completing the construction of linear equations, teachers guide students to propose through analogy: "Can a similar method be used to represent a circle?" Students review the geometric characteristics of a circle and put forward two core attributes: (1) center (fixed point), (2) radius (fixed length). Teachers further guide students to ask: "How to describe 'the distance from any point on the circle to the center is equal to the radius' in algebraic language?" With the help of the Pythagorean theorem, students naturally derive the standard equation of a circle:

$$(x - a)^2 + (y - b)^2 = r^2 \tag{3}$$

Through this process, students not only master the algebraic representation of a circle but also experience the unity and power of "combination of numbers and shapes" in analytic geometry. Teachers further guide students to compare the similarities and differences in the algebraic representation of lines and circles, deepening their understanding of the methodology of "algebraization of geometric objects" and laying a foundation for the subsequent study of conic sections and other content.

3. Discussion and Reflection

From the perspective of problem posing, this study focuses on the teaching design of chapter opening lessons in high school mathematics, taking "Equations of Lines and Circles" as an example, and explores a teaching path integrating historical context, cultural comparison, interdisciplinary connection, and internal knowledge analogy. The study for the first time systematically constructs a four-dimensional questioning scaffold of "time-space-external domain-internal domain", transforming the core problems of chapter opening lessons into operable teaching practices, and breaking through the limitations of knowledge listing and one-way indoctrination in traditional teaching.

The study further draws on the "What-If-Not" method to guide students to independently construct the equations of lines and circles through the algebraic negation of geometric attributes, realizing the transformation of teaching paradigm from "knowledge telling" to "problem generation" and from "passive acceptance" to "active inquiry". This process not only deepens students' understanding of the "combination of numbers and shapes" thought in analytic geometry but also effectively improves their ability to discover and propose problems, responding to the requirements of core competencies and structured teaching in the Curriculum Standards (2022 version).

Tao Xingzhi once said: "Action is the beginning of knowledge; knowledge is the achievement of action." This study is a practical annotation of this concept. Through problem posing activities, students explore the origin of mathematics in "action" and construct the system of knowledge in "knowledge". In the future, we look forward to extending this teaching framework to more mathematical units, helping students become active learners who "dare to ask questions, can explore, and are capable of creating". As Descartes said: "I think, therefore I am." Only in the continuous negation and reconstruction of thinking can mathematics education truly ignite the light in students' hearts and illuminate their future with infinite possibilities.

Funding: This work was financially supported by the Postgraduate Research & Practice Innovation Program of Jiangsu Province (Grant No. KYCX25_1874).

References

1. E. A. Silver, and Y. Li, "Cross-national perspectives on mathematical problem posing," *Asian Journal for Mathematics Education*, 2025. doi: 10.1177/27527263251347663
2. M. A. Icbay, "Accomplishing lesson ending: bringing lesson to an end," *Classroom Discourse*, vol. 16, no. 1, pp. 87-115, 2025. doi: 10.1080/19463014.2024.2407791
3. M. J. Schleppegrell, and B. Bowman, "Problem-posing: A tool for curriculum renewal," *ELT Journal*, vol. 49, no. 4, pp. 297-306, 1995.
4. I. Kupchenko, and J. Parsons, "Ways of Teaching Values: An Outline of Six Values Approaches," 1987.
5. B. M. Kinach, "Understanding and Learning-to-explain by Representing Mathematics: Epistemological Dilemmas Facing Teacher Educators in the Secondary Mathematics``Methods''Course," *Journal of Mathematics Teacher Education*, vol. 5, no. 2, pp. 153-186, 2002.
6. J. Cai, "Integrating problem posing into school mathematics," *Asian Journal for Mathematics Education*, 2025. doi: 10.1177/27527263251346990
7. M. Johansson, "Textbooks in mathematics education: A study of textbooks as the potentially implemented curriculum (Doctoral dissertation, Lulea tekniska universitet)," 2003.
8. S. I. Brown, and M. I. Walter, "The art of problem posing," *Psychology Press*, 2005. doi: 10.4324/9781410611833

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of the publisher and/or the editor(s). The publisher and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.