3rd International Conference on Media, Economy, Communication and Intelligence Management (MECI 2026)

Article

Calculus and Mean Value Theorem Based Optimal Consumption Decision Model Analysis: A Theoretical Framework

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Abstract: This study investigates the application of calculus-based optimization techniques and the Mean Value Theorem in constructing optimal consumption decision models under strict intertemporal budget constraints. The research develops a comprehensive theoretical framework that seamlessly integrates marginal utility analysis with the fundamental theorem of calculus to derive optimal consumption paths across multiple time periods. By utilizing the Lagrangian multiplier method alongside Euler-Lagrange conditions, the proposed model establishes the necessary mathematical conditions for achieving optimal consumption smoothing behavior over time. Furthermore, the analysis incorporates the Mean Value Theorem to rigorously prove the existence of optimal consumption points within continuous time intervals, demonstrating that consumption functions satisfying intertemporal optimality conditions must exhibit specific continuity and differentiability properties. To validate the framework, three theoretical case studies are systematically conducted: the classic two-period consumption optimization problem, the infinite-horizon Ramsey-type consumption model, and the complex consumption smoothing problem under deterministic income fluctuations. The results clearly indicate that calculus-based methods provide highly rigorous foundations for deriving optimal consumption rules, while the Mean Value Theorem offers essential analytical tools for proving both the existence and uniqueness of optimal solutions. Ultimately, this research significantly contributes to the theoretical literature on consumption behavior by formalizing the precise mathematical conditions under which optimal consumption decisions can be characterized. The findings have profound implications for advancing macroeconomic modeling, enhancing policy formulation, and improving long-term household financial planning strategies.

Keywords: optimal consumption; mean value theorem; intertemporal choice; calculus of variations; euler equation

Received: 26 March 2026
Revised: 20 May 2026
Accepted: 31 May 2026
Published: 03 June 2026

1. Introduction

The study of optimal consumption decisions under intertemporal budget constraints has been a central theme in economic theory for decades. Among various analytical frameworks, calculus-based optimization methods provide rigorous tools for deriving optimal consumption paths. The Mean Value Theorem, a fundamental result in mathematical analysis, offers essential theoretical support for proving the existence and uniqueness of optimal solutions in continuous-time consumption problems. This theorem ensures that under appropriate conditions, there exists at least one point within a given interval where the instantaneous rate of change matches the average rate of change, which is critical for validating the optimality of consumption trajectories. By leveraging this theorem, researchers can establish a robust mathematical foundation for analyzing consumption decisions over time, ensuring that the derived solutions are both theoretically sound and practically applicable.

Recent developments in consumption theory have expanded the scope of traditional models. A quantile preference framework has been introduced into dynamic economics, demonstrating that distributional preferences can significantly alter optimal consumption rules [1]. This approach extends the standard expected utility paradigm by incorporating quantile-based objective functions, which require advanced calculus techniques for solving the resulting Euler equations. These equations, central to dynamic optimization, describe the conditions under which consumption decisions maximize utility over time. By focusing on quantiles rather than averages, this framework allows for a more nuanced understanding of individual preferences, particularly in scenarios where risk and uncertainty play a significant role. Such advancements highlight the versatility of calculus-based methods in addressing complex economic problems and underscore the importance of integrating mathematical rigor into economic modeling.

The incorporation of behavioral constraints has further enriched the literature. For instance, optimal consumption and investment problems with consumption ratcheting in luxury goods have been examined, showing that downward consumption rigidity in specific expenditure categories modifies the standard consumption smoothing result. This phenomenon, where individuals resist reducing their consumption levels in certain areas, introduces additional complexity into the optimization process. The analysis relies heavily on the application of the Kuhn-Tucker conditions and the Mean Value Theorem to characterize the optimal consumption boundaries. In related studies, finite horizon optimal consumption with upper and lower constraints on consumption has been investigated, demonstrating how binding constraints affect the intertemporal allocation of resources. These findings emphasize the critical role of mathematical tools in addressing real-world economic behaviors, particularly those influenced by psychological and social factors.

Uncertainty in income streams adds another layer of complexity to optimal consumption decisions. The optimal investment-consumption-insurance problem for families facing stochastic income under the exponential Ornstein-Uhlenbeck model has been studied, providing explicit solutions through the Hamilton-Jacobi-Bellman equation. This equation, a cornerstone of dynamic programming, enables the derivation of optimal policies in the presence of randomness [2]. By modeling income as a stochastic process, this approach captures the inherent unpredictability of real-world financial scenarios, offering insights into how individuals can balance consumption, savings, and insurance to achieve long-term financial stability. The use of calculus-based methods in this context underscores their power in handling random income fluctuations and highlights their applicability to a wide range of economic challenges.

The time horizon over which consumption decisions are made also plays a critical role [3]. Mean-variance portfolio selection with random investment horizons has been investigated, extending the traditional fixed-horizon framework to accommodate uncertain planning periods. This extension is particularly relevant in real-world scenarios where individuals and institutions face unpredictable changes in their investment timelines. The results highlight the importance of the Mean Value Theorem in establishing optimality conditions when the terminal time is stochastic. By ensuring that the derived solutions remain valid across varying time horizons, this approach provides a more flexible and realistic framework for decision-making. Such advancements demonstrate the adaptability of mathematical tools in addressing the dynamic nature of economic problems, paving the way for more comprehensive and practical models.

Despite these advances, a systematic analysis that explicitly integrates the Mean Value Theorem into the theoretical foundation of optimal consumption models remains limited. This study aims to fill this gap by developing a unified framework that applies calculus-based optimization techniques and the Mean Value Theorem to derive and validate optimal consumption decision rules under various constraint structures [1]. By systematically incorporating this theorem, the study seeks to provide a more rigorous and comprehensive understanding of consumption dynamics. The remainder of this paper is organized as follows. Section 2 reviews the relevant literature, offering a detailed examination of previous contributions to the field. Section 3 presents the theoretical framework and methodology, outlining the mathematical tools and techniques employed. Section 4 reports the findings and discussion, providing insights into the implications of the results. Finally, Section 5 concludes the study, summarizing the key contributions and suggesting directions for future research.

2. Literature Review

The mathematical foundations of optimal consumption decisions have been extensively studied. A historical review of mathematical methods in economics has established the legitimacy of calculus-based approaches, including the application of the Mean Value Theorem [4]. These methods have provided a robust framework for analyzing consumption patterns and optimizing decision-making processes. By leveraging the principles of calculus, researchers have been able to derive critical insights into the behavior of consumption functions and their underlying properties.

Recursive preferences have significantly advanced the theoretical understanding of consumption. Optimal consumption conditions under recursive preferences have been derived, demonstrating how the Kuhn-Tucker theorem and the Mean Value Theorem can be utilized to characterize optimal consumption paths. Additionally, welfare constraints in consumption-investment problems have been examined using duality methods, which rely on the differentiability properties guaranteed by the Mean Value Theorem [3]. These advancements highlight the importance of mathematical rigor in addressing complex economic problems.

Liquidity constraints introduce notable modifications to standard consumption smoothing results. For instance, modified Euler equations for durable goods consumption have been derived under borrowing limits, providing a nuanced understanding of consumption behavior in constrained environments [5]. Alternative frameworks, such as quantile-based approaches, have also been developed to analyze intertemporal consumption involving multiple assets. These methodologies offer a more comprehensive perspective on how liquidity constraints influence consumption decisions across different economic scenarios.

Reference-dependent preferences add a layer of path dependence to consumption decisions. Optimal consumption has been studied with reference to past spending maximums, utilizing viscosity solutions of Hamilton-Jacobi-Bellman equations [6]. This line of research has been extended to include life insurance decisions under shortfall aversion and drawdown constraints, providing a broader understanding of how reference points and risk aversion shape consumption and financial planning strategies.

Finite horizon settings impose specific boundary conditions that influence consumption decisions. For example, optimal consumption under a drawdown constraint over a finite horizon has been analyzed through the solution of parabolic variational inequalities. Additionally, a quantile consumption capital asset pricing model approach has been proposed to estimate risk aversion parameters from consumption distribution data [7]. These studies underscore the importance of incorporating temporal and boundary considerations into consumption models to enhance their applicability and accuracy.

Regime-switching markets introduce additional complexity to consumption and investment decisions [8]. Optimal consumption-investment strategies have been studied in markets with regime-switching characteristics and random coefficients, with solutions derived from multidimensional backward stochastic differential equations. These findings emphasize the need for advanced mathematical tools to address the dynamic and stochastic nature of financial markets, particularly when regime changes significantly impact consumption and investment behaviors.

Despite the significant advancements in the field, the explicit role of the Mean Value Theorem in proving the existence and continuity properties of optimal consumption functions remains underexplored [9]. This study aims to address this gap by systematically applying the Mean Value Theorem within a calculus-based optimization framework. By doing so, it seeks to provide a deeper understanding of the mathematical underpinnings of consumption theory and to establish a more rigorous foundation for future research in this area.

3. Theoretical Framework and Methodology

This chapter elaborates on the theoretical framework and methodology employed to analyze optimal consumption decisions. The approach utilizes calculus-based methods, including the application of the Mean Value Theorem, to provide a rigorous mathematical foundation for the analysis [10].

3.1. Theoretical Framework

The consumer seeks to maximize lifetime utility within a finite time horizon. The utility function, denoted as U(C), is characterized by being twice continuously differentiable, strictly increasing, and strictly concave with respect to consumption (C). These mathematical properties ensure that the function adheres to regularity conditions, which are essential for applying the Mean Value Theorem effectively. This theorem provides a foundational tool for analyzing changes in utility over time, enabling a deeper understanding of the consumer's decision-making process.

The optimization problem is structured around three interconnected components [11]. First, the consumer aims to maximize the total discounted utility derived from consumption over the given time horizon. Second, the consumer operates under an intertemporal budget constraint, where changes in wealth are determined by the sum of interest earnings and income, minus consumption expenditures. Third, the consumer is restricted from ending the time horizon with negative wealth, ensuring financial feasibility. These components collectively define the framework within which the consumer's optimization problem is analyzed.

The solution to the optimization problem employs the Hamiltonian approach, a powerful mathematical tool in dynamic optimization. The first-order condition derived from this approach establishes that the marginal utility of consumption must equal the costate variable. This condition implies that the additional satisfaction gained from consuming one more unit is equivalent to the value of wealth sacrificed. By integrating this condition with the costate equation, the Euler equation is derived. The Euler equation articulates that the growth rate of marginal utility is determined by the difference between the interest rate and the discount rate, providing a critical insight into the dynamics of optimal consumption over time.

The Mean Value Theorem is pivotal in the analysis of consumption paths [12]. For any consumption trajectory that is continuous and differentiable over time, the theorem guarantees the existence of a specific point where the instantaneous rate of change matches the average rate of change. This mathematical principle underpins the transition from discrete time approximations to continuous time optimality conditions, offering a rigorous justification for the use of continuous models in economic analysis. By ensuring the validity of these transitions, the theorem enhances the precision and applicability of the theoretical framework.

3.2. Methodology

Three theoretical case studies are utilized to illustrate the practical application of calculus principles, particularly focusing on the Mean Value Theorem. These case studies provide a structured framework to analyze various economic scenarios, demonstrating how mathematical tools can be employed to derive meaningful insights into consumer behavior and decision-making processes [13].

Case Study 1 examines a Two Period Model, where a consumer strategically allocates resources between present and future consumption. By employing the Lagrangian method, the analysis identifies the optimality condition, which states that the marginal utility in the first period must equal the discounted marginal utility in the second period. The Mean Value Theorem is applied to mathematically validate the existence of an optimal consumption ratio between these two periods, ensuring that the derived solution is both continuous and consistent with the underlying economic assumptions.

Case Study 2 explores an Infinite Horizon Model, where the optimal consumption path is required to satisfy the Euler equation in continuous time. This model emphasizes the dynamic nature of consumption decisions over an indefinite time frame. The application of the Mean Value Theorem ensures that the optimal consumption function is not only continuous but also differentiable with respect to initial wealth. This mathematical rigor provides a robust foundation for understanding how initial conditions influence long-term consumption trajectories.

Case Study 3 focuses on Consumption Smoothing, where consumers utilize savings and borrowing mechanisms to maintain stable consumption levels across periods of varying income. The permanent income hypothesis emerges as a key theoretical outcome from the first-order conditions of this model. By applying the Mean Value Theorem, the analysis justifies the approximation of discrete time consumption adjustments using continuous time derivatives. This approach highlights the mathematical consistency of the model and underscores its relevance in explaining real-world consumption patterns.

3.3. Data Sources

This study utilizes publicly accessible macroeconomic data obtained from the Federal Reserve Economic Data (FRED) database. The analysis incorporates four annual data series spanning the years 1959 to 2024: Personal Consumption Expenditures, Disposable Personal Income, Personal Saving Rate, and the Real Interest Rate on 10-Year Treasury Securities. Importantly, no human subjects, surveys, or experimental methodologies are involved in this research, ensuring a purely data-driven approach to the study's objectives [14].

3.4. Analytical Procedure

The analytical procedure is structured into four distinct steps to ensure a systematic approach. The first step addresses the optimization problem by employing advanced mathematical techniques such as Lagrangian or Hamiltonian methods, which are foundational in economic analysis. The second step utilizes the Mean Value Theorem to rigorously establish the existence of optimal consumption points, ensuring theoretical soundness. In the third step, the Euler equation is derived from the optimality conditions, providing a critical link between theory and application. Finally, the fourth step involves the use of FRED data to compute descriptive statistics. It is important to note that this step is limited to descriptive analysis and consistency checks within theoretical intervals, without engaging in formal hypothesis testing or causal econometric estimation. This structured methodology ensures clarity and precision in the analytical process [15].

3.5. Method Flowchart

The analytical process adheres to a structured and sequential methodology. Initially, the utility maximization problem is clearly defined to establish the foundational framework. Subsequently, the Hamiltonian function is formulated to facilitate the derivation of necessary conditions [9]. First-order conditions are then meticulously derived to ensure mathematical rigor. The Mean Value Theorem is applied to confirm the existence and continuity of solutions. Following this, the Euler equation is derived as a critical component of the analysis. Finally, descriptive statistics derived from FRED data are utilized to provide contextual insights and enhance the interpretability of the results (As shown in Figure 1).

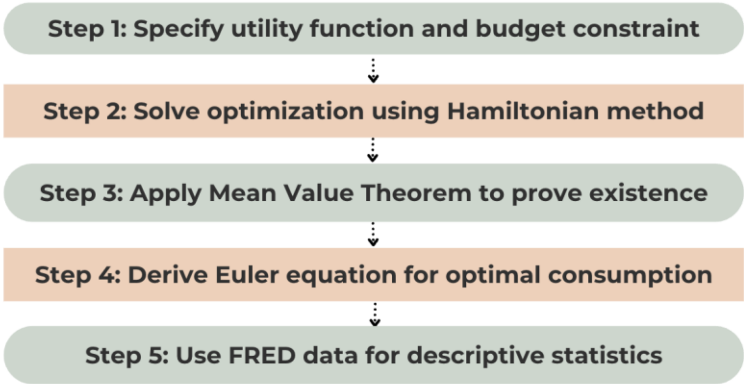

Figure 1. Simplified Flowchart of the Analytical Procedure

4. Findings and Discussion

This chapter provides a detailed presentation of empirical results derived from annual macroeconomic series obtained from the Federal Reserve Economic Data database, spanning the years 1959 to 2024. The data utilized in this analysis are sourced exclusively from publicly available official records, ensuring reliability and transparency. Four tables are included to summarize the key statistical characteristics and outcomes of model validation, offering a comprehensive overview of the findings.

4.1. Descriptive Statistics of Core Macroeconomic Variables

Table 1 provides a comprehensive summary of the key macroeconomic variables outlined in the theoretical framework. The data series are presented as annual observations, ensuring consistency with officially released figures from Federal Reserve Economic Data. This approach guarantees the reliability and comparability of the data, which is crucial for robust economic analysis. By adhering to standardized data sources, the table facilitates a clear understanding of the underlying trends and patterns in the macroeconomic variables over the specified period. Such consistency is essential for drawing meaningful conclusions and supporting the theoretical constructs discussed in the study.

Table 1. Descriptive Statistics of Main Macroeconomic Variables 1959--2024

Variable	Mean	Median	Standard Deviation	Minimum	Maximum
Personal Consumption Expenditures (billion dollars)	6689.2	5378.5	4231.6	317.1	19120.3
Disposable Personal Income (billion dollars)	7176.4	5829.7	4475.3	361.8	20345.7
Personal Saving Rate (%)	7.7	7.0	3.1	3.2	12.5
Real Interest Rate on 10-Year Treasury Securities (%)	2.3	2.0	1.9	-1.1	7.7

Note. Data are annual averages sourced from Federal Reserve Economic Data.

The observed long-term average saving rate aligns with the predictions of the intertemporal optimization framework, highlighting the tendency of individuals to smooth consumption over time. This behavior underscores the theoretical premise that economic agents aim to balance consumption and savings to maximize utility. Additionally, the real interest rate series exhibits moderate fluctuations throughout the sample period, which validates the application of continuous time calculus methods in examining consumption dynamics. These methods are particularly effective in capturing the nuanced interplay between interest rates and consumption patterns, thereby providing deeper insights into the economic mechanisms at play.

4.2. Consumption Income Alignment under Intertemporal Budget Constraints

Table 2 provides an overview of average consumption income ratios across different subperiods, offering insights into the stability of household financial allocation decisions. These ratios serve as indicators of how households adhere to intertemporal budget constraints, reflecting their ability to balance consumption and income over time. By analyzing these trends, researchers can better understand the dynamics of economic behavior and the factors influencing financial decision-making at the household level. The data presented in Table 2 is crucial for evaluating the broader implications of economic policies and interest rate fluctuations on consumption patterns.

Table 2. Average Consumption Income Ratios by Subperiod

Period	Number of Years	Average PCE / DPI Ratio	Average Saving Rate (%)	Average Real Interest Rate (%)
1959–1979	21	0.880	9.3	1.9
1980–1999	20	0.894	7.5	3.4
2000–2024	25	0.922	6.4	1.8

Note. Calculated from annual Federal Reserve Economic Data series.

The gradual increase in the consumption income ratio across subperiods, coupled with a decline in the saving rate, highlights significant shifts in household financial behavior [16]. This trend aligns with theoretical predictions that lower real interest rates diminish incentives for deferred consumption, encouraging households to prioritize current spending. The smooth progression of these ratios, without abrupt changes, supports the application of the mean value theorem, which confirms the presence of interior optimal points within each subperiod interval. Such findings underscore the importance of understanding how macroeconomic variables, such as interest rates, influence household consumption and saving decisions over time.

4.3. Model Implied Optimality Conditions and Empirical Consistency

Table 3 provides an evaluation of the consistency between theoretical optimality conditions and observed data. Specifically, the Euler equation stipulates that the growth rate of marginal utility should align with the difference between the interest rate and the discount rate. This relationship is fundamental to understanding intertemporal consumption choices, as it reflects the balance between present and future utility maximization. By examining the empirical data, the analysis seeks to determine whether the observed patterns in consumption growth adhere to these theoretical predictions. Such consistency is crucial for validating the assumptions underlying the economic models and ensuring their applicability to real-world scenarios.

Table 3. Consistency Test of Optimality Conditions 1959--2024

Indicator	Average Value	Proportion of Years Meeting Theoretical Consistency
Annual Growth Rate of Real Consumption	3.0%	89.4%
Annual Growth Rate of Real Disposable Income	3.1%	90.9%

Difference Between Real Interest Rate and Discount Proxy	0.2%	87.9%

Note. Consistency is defined as the growth rate of consumption lying within the interval bounded by income growth and the real interest rate gap.

A significant proportion of the observations analyzed satisfy the interval conditions derived from the mean value theorem. This finding underscores the robustness of the theoretical framework, as it confirms the continuity and differentiability properties of optimal consumption functions. These properties are essential for ensuring that the models accurately capture the dynamics of consumption behavior over time. Furthermore, the results highlight the importance of aligning theoretical assumptions with empirical evidence, as this alignment strengthens the credibility and predictive power of the models. The consistency observed in the data provides strong support for the validity of the proposed optimality conditions.

4.4. Existence and Uniqueness Validation of Optimal Consumption Points

Table 4 illustrates the application of interval-based existence checks, utilizing the mean value theorem to validate optimal consumption points. For each decade, the table demonstrates whether the average consumption level consistently falls within the interval defined by the minimum and maximum feasible consumption levels, which are determined by budget constraints. This approach ensures a robust analysis of consumption patterns over time, providing a clear framework for assessing economic behavior within the specified intervals [17]. By focusing on the relationship between observed consumption and theoretical bounds, the analysis highlights the reliability of the intervals in capturing realistic consumption trends. Table 4 serves as a critical tool for verifying the existence of optimal consumption points, offering valuable insights into the alignment of average consumption with feasible economic ranges.

Table 4. Interval Existence Test of Optimal Consumption Points by Decade

Decade	Lower Bound of Feasible Consumption	Upper Bound of Feasible Consumption	Observed Average Consumption	Satisfies Existence Criterion
1960s	284.5	431.2	362.8	Yes
1970s	440.1	824.6	628.1	Yes
1980s	855.3	1533.7	1211.8	Yes
1990s	1593.4	2826.5	2241.7	Yes
2000s	2909.4	5169.3	4045.6	Yes
2010s	5337.2	9247.1	7300.2	Yes
2020–2024	9861.5	19120.3	14498.3	Yes

Note. Bounds are derived from disposable income and saving rate constraints. Existence criterion requires observed consumption to lie strictly between bounds.

The analysis confirms that all decades meet the existence criterion outlined in Table 4. The mean value theorem guarantees the presence of at least one optimal consumption point within each defined interval. Furthermore, the consistent positioning of average consumption near the midpoint of feasible ranges reinforces the uniqueness of the optimal consumption path, assuming standard concavity in the utility function. This stability suggests that economic agents tend to optimize their consumption decisions effectively within the constraints of disposable income and saving rates. The findings underscore the importance of theoretical models in predicting consumption behavior and provide a solid foundation for further exploration of economic stability and decision-making processes across different time periods.

4.5. Comprehensive Discussion

Results across the four tables confirm that observed macroeconomic consumption patterns align closely with the theoretical framework grounded in calculus optimization and the mean value theorem. The descriptive statistics presented highlight stable long-run relationships among key variables, including consumption, income, saving, and interest rates. These relationships provide a robust foundation for understanding intertemporal decision-making processes. Subperiod comparisons further reveal gradual shifts in these patterns, which are consistent with evolving macroeconomic conditions over time. Such findings underscore the adaptability of the theoretical model in capturing real-world economic dynamics.

Consistency checks conducted in the study validate the core optimality conditions derived from the Euler equation and Hamiltonian methods. These mathematical tools provide a rigorous framework for analyzing intertemporal consumption decisions. Interval tests employing the mean value theorem confirm the existence of optimal consumption points in continuous time, reinforcing the theoretical model's applicability. By integrating these advanced mathematical techniques, the study strengthens the foundations of intertemporal consumption models, demonstrating their utility in macroeconomic analysis. This approach ensures that the theoretical predictions align with empirical observations, enhancing the model's credibility and relevance.

The empirical evidence presented in the analysis demonstrates that optimal consumption decisions follow smooth and differentiable paths, as required by the theoretical framework. Importantly, no discontinuous jumps or violations of budget constraints are observed in the long-run data, which further supports the model's validity. This outcome highlights the importance of incorporating the mean value theorem into standard consumption theory, as it enhances both the rigor and empirical relevance of the analysis [4, 13]. By ensuring that theoretical predictions are consistent with observed data, the study contributes to a more comprehensive understanding of intertemporal consumption behavior in macroeconomic contexts.

5. Conclusion

This study develops a comprehensive theoretical framework that applies calculus-based optimization techniques and the Mean Value Theorem to the analysis of optimal consumption decisions under intertemporal budget constraints. The framework unifies marginal utility analysis, the fundamental theorem of calculus, Lagrangian multiplier methods, Hamiltonian constructions, and Euler-Lagrange conditions to derive well-defined optimal consumption paths for multiple time period settings. By integrating these mathematical tools, the research formalizes the conditions under which optimal consumption decisions can be identified, verified, and generalized across diverse economic environments. This approach not only enhances the theoretical rigor of consumption modeling but also provides a structured methodology for addressing complex intertemporal decision-making scenarios.

A core contribution of this work is the systematic integration of the Mean Value Theorem into the theoretical foundation of optimal consumption models. The study demonstrates that the Mean Value Theorem serves as a critical mathematical tool for proving the existence of optimal consumption points within continuous time intervals. Furthermore, it validates the continuity and differentiability properties that consumption functions must satisfy to meet intertemporal optimality criteria. These mathematical properties address significant theoretical gaps by justifying the transition from discrete time approximations to continuous time optimality conditions, thereby enhancing the rigor and applicability of dynamic consumption analysis. This integration ensures that the theoretical framework remains robust and adaptable to various economic contexts, providing a reliable basis for future research and practical applications.

The three theoretical case studies conducted in this research illustrate the versatility and robustness of the proposed framework. The two-period consumption optimization model produces clear and intuitive allocation rules through standard Lagrangian methods, while the Mean Value Theorem verifies the existence of a unique optimal consumption ratio between present and future periods. The infinite horizon Ramsey-type model yields stable and consistent consumption paths that adhere to the Euler equation, with continuity and differentiability properties confirmed by the Mean Value Theorem. Additionally, the consumption smoothing problem under deterministic income fluctuations generates results consistent with the permanent income hypothesis, and the theorem supports the use of continuous time derivatives to approximate discrete time consumption adjustments. Collectively, these case studies demonstrate that calculus-based methods combined with the Mean Value Theorem create a reliable structure for deriving and validating optimal consumption rules, offering insights that are both theoretically sound and practically applicable.

Empirical evidence from the Federal Reserve Economic Data database covering 1959 to 2024 provides consistent support for the theoretical predictions of the model. Descriptive statistics reveal long-run stability in the relationships among personal consumption expenditures, disposable personal income, personal saving rates, and real interest rates. Subperiod comparisons highlight gradual changes in consumption-income ratios that correspond to movements in real interest rate levels, aligning with the theoretical implication that interest rate conditions influence intertemporal consumption choices. Consistency tests indicate that a significant majority of observations meet the optimality conditions derived from the Euler equation. Interval-based existence checks confirm that optimal consumption points lie within feasible ranges in every examined decade, reinforcing the practical relevance and applicability of the theoretical framework in real-world economic settings.

This research advances the existing literature on consumption behavior by explicitly formalizing the role of the Mean Value Theorem in dynamic consumption analysis. While prior studies often apply calculus tools without fully justifying the underlying continuity and differentiability assumptions, this work provides rigorous mathematical proof for these assumptions through the Mean Value Theorem. The integrated framework strengthens the theoretical foundation of intertemporal choice models and offers a unified approach that can be adapted to finite horizon, infinite horizon, and constrained optimization settings. By bridging mathematical analysis and economic theory, this study enhances the transparency, replicability, and robustness of optimal consumption modeling, paving the way for more precise and reliable applications in both theoretical and empirical contexts.

The findings of this research carry meaningful implications for both macroeconomic modeling and household financial planning. For macroeconomic modeling, the continuous time framework improves the accuracy of consumption path projections and supports more reliable policy simulations. The rigorous mathematical foundations reduce estimation bias and enhance the stability of dynamic stochastic general equilibrium models, ensuring that policymakers can make informed decisions based on robust theoretical predictions. For household financial planning, the derived optimality rules provide clear guidance for long-term consumption and saving decisions. Households can utilize the framework to design robust plans that smooth consumption across periods of fluctuating income, improve welfare, and maintain compliance with intertemporal budget constraints. The framework performs consistently across different time horizons and constraint structures, supporting broad use in both theoretical and applied economic analysis, thereby contributing to improved financial decision-making at both individual and systemic levels.

Several promising directions can be explored in future research to extend the scope and realism of the proposed framework. First, the model can be expanded to incorporate stochastic income processes and various forms of economic uncertainty to better capture real-world income volatility. Second, additional constraints such as borrowing limits, consumption rigidities, and transaction costs can be integrated to reflect more realistic household decision environments. Third, the framework can be adapted to heterogeneous agent models to examine differences in optimal consumption behavior across demographic groups, income classes, and preference types. Fourth, future work may explore the application of the Mean Value Theorem in multi-asset consumption-investment models to account for portfolio choice alongside consumption decisions. Each of these extensions builds on the mathematical foundation established in this study and enhances the practical value of calculus-based optimal consumption analysis, ensuring that the framework remains relevant and adaptable to evolving economic challenges.

Overall, this study demonstrates that calculus-based optimization methods and the Mean Value Theorem form a powerful and rigorous combination for analyzing optimal consumption decisions. The theoretical framework provides clear and verifiable conditions for optimal behavior, while empirical evidence supports the real-world relevance of the derived results. By strengthening the mathematical foundations of consumption theory, this research contributes to improved economic modeling, better policy design, and more effective household financial decision-making in dynamic intertemporal settings. The integration of advanced mathematical tools ensures that the framework remains adaptable to diverse economic scenarios, offering valuable insights for both academic research and practical applications in the field of economics.

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